About the Schwarzschild-Spacetime

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1 Abstract

The purpose of this article is to give some geometric insight into the Solar System as a Schwarzschild-Spacetime [1]. Schwarzschild-Spacetime is in the sense of general theory of relativity a Riemann-manifold with a Schwarzschild-Metric. The geometry of the geodesics will be described applying mathematical methods.
In the Schwarzschild-Spacetime all geodesics with bound orbit lie in one and only one plane will be proofed by study of two geodesics with bound orbit.

2 Introduction

The main subject of the General Theory of Relativity (GTR) is the so called Spacetime. The GTR looks at the universe as a special Spacetime. The Spacetime is a 4-dimensional manifold with a pseudo-Riemannian metric. The Einstein field equation connects the curvature of this manifold's with the energy-momentum tensor. For solving the Einstein-Field-Equation there are known several different metrics. On the other side these different metrics can be seen as manifolds with different curvature. The simplest form is a full spherical symmetry. This metric is called the Schwarzschild-Metric and the corresponding space- time - the Schwarzschild-Spacetime.
The Schwarzschild-Spacetime is a good interpretation of our solar system. Looking at our solar system, it is striking that all planets orbit the Sun in nearly one and only one plane. We will present that this is a property of the Schwarzschild-Spacetime.
First, a short introduction of terms.
Definition 1 (Manifold). An n-dimensional k differentiable manifold is a topological space M in which each point $x_i \in M$ has a neighborhood $U_i \subset M$ and a homeomorphism $\varphi_i : U_i \to V_i \subset \mathbf{R}^n$ into the open set $V_i \subset \mathbf{R}^n$ such that

$$\varphi_i \circ \varphi_j^{-1} : \ \varphi_j(U_i \cap U_j) \to {\varphi_i}^{-1}(U_i \cap U_j)$$

(between open sets in $\mathbf{R}^n$) is k-differentiable.
□
Since $\varphi_i : U_i \to V_i$ is a map into $\mathbf{R}^n$, φ_i has the form

$$\varphi_i(x) = (\varphi_i^1(x), \ldots, \varphi_i^n(x)), \quad x \in M$$

Therefore, the homeomorphism φ_i is called a coordinate system, the system of $\{\varphi_i, U_i\}$ a chart system and $\varphi_i \circ \varphi_j^{-1}$ a coordinate transformation.

The tangent bundle TM of a differentiable manifold M is the disjoint union of the tangent spaces $T_PM, \quad P \in M$:

$$TM = \cup_{P\in M} T_PM = \cup_{P\in M}\{P\} \times T_PM.$$

The tangent space T_PM at $P \in M$ can be considered as an $\mathbf{R}^n$. More exactly, the $\mathbf{R}^n$ is the set of all tangential vectors of all curves on M through the point $P \in M$.
The manifold is called metric, if in each tangent space is given a metric, such, that the metric is invariant by coordinate transitions.
Defintion 2: (Pseudo-Riemannian Manifold). A differentiable manifold M is a pseudo-Riemannian manifold, if for each point $P \in M$ a tensorial, symmetric and non degenerate map $g_p : T_P(M) \times T_P(M) \rightarrow \mathbf{R}$ is given such that for each $X, Y \in T_P(M), a, b \in C^\infty(M)$

$$1. \quad g_P(aX + bY, Z) = a(P)g_P(X, Z) + b(P)g_P(Y, Z) \quad (tensorial) \tag{1}$$
$$2. \quad g_P(X, Y) = g_P(Y, X) \quad (symmetrical) \tag{2}$$
$$3. \quad there\ is\ no\ X \in T_P(M),\ so\ that\ g_P(X, Y) = 0\ for\ all\ Y. \tag{3}$$

□
Applying the bilinear form g_p to an orthogonal basis of $T_P(M)$, we obtain n values which are positive, negative or equal zero, independent of the selected orthogonal basis. This sign combination is called **signature of the metric**. The condition 3. of the pseudo- Riemannian metric means, that the signature value 0 does not exist. The special case of the signature is $(1, n-1)$ (or $(n-1, 1)$), then M is called a **Lorentzian manifold**.
The dual space to the tangent space $T_P(M)$ is the space of all differential forms. So the pseudo-Riemannian metric is a 2-differential form.
We assign the local coordinates of the Spacetime as

$$x_0, x_1, x_2, x_3 \quad where\ x_0 = t \in \mathbf{R}\ and\ (x_1, x_2, x_3) \in \mathbf{R}^3 \tag{4}$$

The coordinate $x_0 = t$ is the time coordinate and the $(x_1, x_2, x_3) \in \mathbf{R}^3$ the space coordinates. The Schwarzschild-Spacetime is a pseudo-Riemannian manifold (in our case the $\mathbf{R}^4$) with the Schwarzschild-Metric[1]. The Schwarzschild-Spacetime is a universe with only one non rotating gravitation-center.
For the spherical symmetry the most suitable space coordinates are the spherical coordinates $(r, \vartheta, \varphi), r \in \mathbf{R}^+, \vartheta \in [0, \pi], \varphi \in [-\pi, \pi]$. r is the radial distance to the origin, ϑ the polar angle (measured to a fixed zenith direction, the z-axis), and φ the azimuth angle (the angle of its orthogonal projection

to the xy-plane to the x-axis). In this coordinates is $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^+ \times_r S^2$ and the Schwarzschild-Metric is given by

$$\begin{aligned} &ds^2 = -kdt^2 + k^{-1}dr^2 + r^2(d\vartheta^2 + sin^2\vartheta d\varphi^2), \\ &k = k(r) = 1 - \frac{2M}{r},\ M = mass\ of\ the\ center\ of\ gravity. \end{aligned} \tag{5}$$

We have set $c = G = 1$ (see below). This unit system is called geometric and usual in GTR. In the above coordinate system the Schwarzschild-Metric has a singularity at $r = 2M$ (Schwarzschild-Sphere). In our solar system the Schwarzschild-Sphere is located in the sun, it is very small. Physically, in the Schwarzschild-Sphere the metric changes the sign.
For simplicity we consider the Schwarzschild-Spacetime just outside the Schwarzschild-Sphere:

$$\begin{aligned} &U = \mathbf{R} \times \mathbf{R}^+ \times_r S^2 \setminus \{t, r, \vartheta, \varphi) : r = 2M\} \\ &or \\ &U = (\mathbf{R} \times (0, 2M) \times_r S^2) \bigcup (\mathbf{R} \times (2M, +\infty) \times_r S^2) \end{aligned} \tag{6}$$

We define a scalar product $\langle x_1, x_2 \rangle$ by:

$$\begin{aligned} \langle x_1, x_2 \rangle &= \langle (t^1, r^1, \vartheta^1, \varphi^1), (t^2, r^2, \vartheta^2, \varphi^2) \rangle \\ &= -kt^1t^2 + k^{-1}r^1r^2 + r^2(\vartheta^1\vartheta^2 + sin^2\vartheta\varphi^1\varphi^2) \end{aligned} \tag{7}$$

The spherical coordinates are obviously orthogonal in this inner product.
Still some words to the geometrical units ($c = G = 1$). Because $c = 1$ the length unit $1m$ is a unit of time too - the time the light needs for $1m$:

$$1second = 2,99792 \cdot 10^8 m. \tag{8}$$

Because $G = 1$ (gravity constant) we have

$$1kg = 7,425 \cdot 10^{-28} m. \tag{9}$$

3 Schwarzschild-Geodesics

A geodesic is the locally shortest distance between two points. Geodesics are forcefree movement of particles and photons. The geodesics of the Schwarzschild-Metric are well known (eg [2], [3]).

Theorem 1 (geodesic) *The geodesics in the Schwarzschild-Metric are given by*

$$\frac{d}{ds}[g_{jj}(\frac{dx_j}{ds})] = \frac{1}{2}\sum_{i=0}^{3}\frac{\partial g_{ii}}{\partial x_j}(\frac{dx_i}{ds})^2, j = 0, ..., 3 \tag{10}$$

where $g_{00} = -k, g_{11} = k^{-1}, g_{22} = r^2, g_{33} = r^2 sin\vartheta$
and $x_0 = t, x_1 = r, x_2 = \vartheta, x_3 = \varphi$.

Proof. The proof is given in source, e.g. [3]. We only remind the proof based on calculation of the Christoffel-Symbols for the Schwarzschild-Metric.
□
The geodesic equations (10) are a system of four differential equations. This equations give by calculation the so called equations of motion:

Theorem 2 (equations of motion) *Let be* γ *a geodesic in* U *(see [2]). Then there are two constants,* L *and* E*, such that:*

$$(a) \quad k\frac{dt}{ds} = E \tag{11}$$

$$(b) \quad r^2 sin^2\vartheta\frac{d\varphi}{ds} = L \tag{12}$$

$$(c) \quad \frac{d}{ds}[r^2\frac{d\vartheta}{ds}] = r^2\sin\vartheta\cos\vartheta(\frac{d\varphi}{ds})^2 \tag{13}$$

□
There exist a pronounced coordinate system for each geodesic of the Schwarzschild-Spacetime such, that the geodesic is placed in the equatorial plane. This coordinate system is obtained by rotation.
Definition 3 (equatorial initiating). A curve $\gamma : I \rightarrow U$ is called equatorial initiating relative to the spherical coordinates ϑ, φ *on* S^2, when

$$\vartheta(0) = \tfrac{\pi}{2} \quad und \quad \tfrac{d\vartheta}{ds}(0) = 0$$

□
After a suitable rotation of the coordinate system each curve is equatorial initiating. Therefore for each geodesics is a coordinate system such, that the geodesic is equatorial initiating.

Theorem 3 (geodesic plane) *Let* γ *be a equatorial initiating geodesic in* U*. Then:*

$$\vartheta \equiv \tfrac{\pi}{2}$$

Therefore each geodesic in the Schwarzschild-Spacetime is positioned in the space part in a plane (through the origin).

Proof. The proof of this property is well known. We remind it briefly here. $\vartheta \equiv \frac{\pi}{2}$ is a solution of (13). Since this solution is equatorial initiating, this solution is a unique (with boundary value) solution.

□

Each geodesic of the Schwarzschild-Spacetime is in the space part flat, i.e. in the space part is located in a plane through the origin. In fact, by a simple rotation we achieve the geodesic is equatorial initiating. Then it follows from Theorem 3, the geodesic is located in the new coordinate system in the equatorial plane. Consequently, in a given coordinate system there are for each geodesic γ such constants $a, b \in \mathbf{R}$, that

$$\gamma \subset \{(t, r, \vartheta, \varphi) : \cos\vartheta = (a\cos\varphi + b\sin\varphi)\sin\vartheta\}. \tag{14}$$

4 Particles of Matter

It is a basic statement of GTR, that the time depends on the movement of the time-measuring clock. The special time, which is an invariant is called proper time. The proper time τ is the flying time in the reference system running with the particle. It is

$$d\tau^2 = -ds^2 \tag{15}$$

Let the Schwarzschild-Spacetime be a model of the Universe. Then the coordinate $x_0 = t$ describe the fact, that the universe can only exist as a system in movement. This movement will be described by the coordinate t. Because the movement is the form of existence of the universe, it is not possible to measure this time t. If we postulate, all force-free movements can pass only along geodesics, the measurable parameter is the distance passed on the geodesics, the proper time. Then (15) means, there is no qualitative difference between a parameterization by the line element or by the proper time.

Already it is noted here a big difference to photons. In the GTR the speed of light is the limit of any speed. Photons move with this. Of course, the photons move only on geodesics. The maximality of the light-speed in the space-part of the Schwarzschild-Spacetime imply the identification of the space-part-distance with the distance of the philosophically movement-part. In other words, on geodesics of photons is $ds^2 = 0$, that means, photons

do not have a proper time. Consequently, the speed of light is not exactly measurable in the Schwarzschild-Spacetime.
We quote the following theorem (see e.g. [2], [3]).

Theorem 4 (equations for particles of matter in proper time) *Let be γ a particle of matter parametrized by the proper time τ, free falling and equatorial initiating particle in U (see (6)). Let be L and E the constants of Theorem 2. Then:*

$$(a) \quad k\frac{dt}{d\tau} = E \tag{16}$$

$$(b) \quad r^2\frac{d\varphi}{d\tau} = L \tag{17}$$

$$(c) \quad \vartheta \equiv \frac{\pi}{2} \tag{18}$$

$$(d) \quad E^2 = (\frac{dr}{d\tau})^2 + (1 + \frac{L^2}{r^2})k(r) \tag{19}$$

□
Both constants E *and* L have physical meanings. The equation (12) is similar to Kepler II (the area of the covered driving beams per unit of time is constant). Thus L is the angular momentum per mass unit, or shortly angular momentum.
For a resting particle is $d\tau = \sqrt{k}dt$ and then

$$mE = \lim_{r\to\infty} \frac{mE}{\sqrt{k}} = \lim_{r\to\infty} m\sqrt{k}\frac{dt}{d\tau} = \lim_{r\to\infty} -\frac{m}{\sqrt{k}}\frac{dt}{d\tau}\langle\partial_t, \partial_t\rangle$$

Therefore E is the energy per mass unit at infinity, shortly energy.
In summary, each geodesic is located in the space part in a plane through the coordinate origin and to each geodesic exist two constants E and L, the energy E and the angular momentum L.
The following is a brief explanation of possible orbits of free falling particles of matter in the Schwarzschild-Spacetime.
Definition 4 (Potential). The effective potential-energy $V(r)$ of the geodesic γ is

$$V(r) = (1 + \frac{L^2}{r^2})k(r) = 1 - \frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3}$$

Thus, the energy equation (19) can be written as

$$E^2 = (\frac{dr}{d\tau})^2 + V(r)$$

$\square$

With the help of the potential $V(r)$ it is easy to classify the possible orbits. Since the following considerations are well known, we give them without proof (full explanation of all possible orbits see [6]).

For our purpose, it is sufficient to classify the orbits by the angular-momentum L. The following cases can be shown:

$$\begin{array}{ll} (a) & L^2 < 12M^2 \\ & \textit{then just crash orbit and crash/escape orbit are possible.} \\ (b) & 12M^2 < L^2 < 16M^2 \\ & \textit{then crash orbit, bound orbit and crash/escape orbit are possible.} \\ (c) & L^2 > 16M^2 \\ & \textit{then crash orbit, bound orbit, flyby orbit,} \\ & \textit{and crash/escapeorbit are possible.} \end{array}$$

In summary, we find that bound orbit are only possible, if $L^2 > 12M^2$.

5 Topology of bound orbits

Let γ be a geodesic of the Schwarzschild-Spacetime, $(t, r, \vartheta, \varphi)$ a selected spherical coordinate system, E the energy of γ and L the angular momentum od γ. Since the geodesic is located in the space part in a plane through the coordinate origin, there are two constants $a, b \in \mathbf{R}$ such that

$$\gamma \subset \{(t, r, \vartheta, \varphi) : \cos\vartheta = (a\cos\varphi + b\sin\varphi)\sin\vartheta\}. \tag{20}$$

Theorem 5 (topology of geodesics) *Let γ be a geodesic in the Schwarzschild-Spacetime. Then is $L = 0$, or γ is in the space part a straight line through the origin, or $\vartheta \equiv \frac{\pi}{2}$.*

Proof. By Theorem 3 in the space part $(\mathbf{R}^+ \times_r S^2)$ is the geodesic located in a plane, that is, there exist such $a, b \in \mathbf{R}$, that (20) is satisfied. If this plane is not identical with $\vartheta \equiv \frac{\pi}{2}$, then

$$a^2 + b^2 \neq 0. \tag{21}$$

We choose ϑ_0 and φ_0 such that:

$$\begin{aligned} \cos\vartheta_0 &= \frac{1}{\sqrt{1 + a^2 + b^2}} \quad & \sin\vartheta_0 &= \sqrt{\frac{a^2 + b^2}{1 + a^2 + b^2}} \\ \sin\varphi_0 &= \frac{b}{\sqrt{a^2 + b^2}} \quad & \cos\varphi_0 &= \frac{a}{\sqrt{a^2 + b^2}} \end{aligned} \tag{22}$$

$$\theta = \vartheta - \vartheta_0, \quad \phi = \varphi - \varphi_0, \quad r = r. \tag{23}$$

From (21) and (22) follows that the geodesic γ is located in the equatorial plane

$$\theta \equiv \frac{\pi}{2}. \tag{24}$$

The coordinate transformation (23) is a composite of rotations, first by an angle φ_0 about the z-axis and then about the y-axis by the angle ϑ_0. The easiest way to verify the equation (24) is to do this in the corresponding Cartesian coordinates. The two rotations will be executed by following matrices :

$$\begin{aligned} A &= \begin{pmatrix} \cos\vartheta_0 & 0 & \sin\vartheta_0 \\ 0 & 1 & 0 \\ -\sin\vartheta_0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi_0 & \sin\varphi_0 & 0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & \cos\vartheta_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi_0\cos\vartheta_0 & \sin\varphi_0\cos\vartheta_0 & \sin\vartheta_0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \\ -\cos\varphi_0\sin\vartheta_0 & -\sin\varphi_0\sin\vartheta_0 & \cos\vartheta_0 \end{pmatrix} \end{aligned} \tag{25}$$

The plane equation in (20) has in the Cartesian coordinates the form $z = ax + by$. According to perform the coordinate transformation (25) and after inserting (22) we obtain for the new z'- coordinate $z' = 0$, in other words (24).

Applying trigonometric formulas and using (23) and (24) we obtain:

$$\begin{aligned} \sin^2\vartheta &= \sin^2(\theta + \vartheta_0) \\ &= (\sin\theta\cos\vartheta_0 + \cos\theta\sin\vartheta_0)^2 \\ &= \frac{1}{1+a^2+b^2}(\sqrt{a^2+b^2}\cos\theta + \sin\theta)^2 \\ &= \frac{1}{1+a^2+b^2} \end{aligned} \tag{26}$$

Since the geodesic γ is located in the plane (20), it follows from (26)

$$1 \;-\; \sin^2\vartheta \;=\; (a\cos\varphi \;+ b\sin\varphi)^2\sin^2\vartheta$$

and thus

$$a^2 + b^2 = (a\cos\varphi + b\sin\varphi)^2 \tag{27}$$

The geodesic γ was parameterized by the line element s. The differentiation of (27) to s, multiplication with $r^2\sin^3\vartheta$ and application of (12) gives

$$\begin{aligned} 0 &= (a\cos\varphi + b\sin\varphi)\sin\vartheta(-a\sin\varphi + b\cos\varphi)r^2\sin^2\vartheta\frac{d\varphi}{ds} \\ &= \cos\vartheta(-a\sin\varphi + b\cos\varphi)L \end{aligned} \tag{28}$$

Multiplication by $\cos\vartheta$ and use of (26) gives:

$$0 = \frac{a^2 + b^2}{1 + a^2 + b^2}(-a\sin\varphi + b\cos\varphi)L \tag{29}$$

This equation is satisfied if $L = 0$, or $(a^2+b^2) = 0$, or $(-a\sin\varphi+b\cos\varphi) \equiv 0$. The geodesic γ satisfies (20). The differentiation of the plane equation by the line element s and then multiplying by $\sin\vartheta$ gives

$$\frac{d\vartheta}{ds} = (a\sin\varphi - b\cos\varphi)\sin^2\vartheta\frac{d\varphi}{ds}. \tag{30}$$

If $(-a\sin\varphi + b\cos\varphi) \equiv 0$, then

$$\begin{aligned}\frac{d\vartheta}{ds} &\equiv 0 \\ \vartheta &\equiv const\end{aligned} \tag{31}$$

This means that the geodesic is located on a cone with the vertex in the origin and at the same time in a plane through the origin. Thus, in the space part the geodesic is a straight line through the coordinate origin.
q.e.d. $\square$
If the Schwarzschild-Spacetime has a geodesic with bound orbit, then the coordinate system can be chosen so that the geodesic is in the equatorial plane. For a second geodesic with bound orbit it will be located in the same plane by theorem 5. In other words, all geodesics with bound orbit are located in one and only one plane.

6 Discussion and Conclusion

Let us take a different view on the reality. The planetary orbits are ellipses, which - so our result - are located in one plane. Ellipses are conic sections with the sun in one focus. So there exist for each planet a theoretical rotation cone outside the solar system such, that the sectional area with the planetary plane is the planetary orbit ellipse. More precisely somewhere outside the solar system is the cone tip and there is a axis of rotation through the apex of the cone. All planetary planes are located in one plane, which means, you can take the cones of different planets all having the same axis of rotation, the same cone tip and the planet orbits are obtained with the same cutting plane. The cones just have different opening angles (apertures). This would also mean that all Apo- and Perikles of the the planets are located on one axis (neglecting the apsidal precession).

One might suspect that the conical tip is placed in the black hole of the Milky Way, and hence the opening angle (aperture) of the (planetary) cone is very small.
In any case, it is true the solar system is not by itself explaneable and a better Spacetime metric for the solar system would have to take into account gravitational centers outside the solar system.
Another conclusion, when we speak of the infinity of the universe, there can be no Spacetime-Metric, describing the universe as spacetime completely (Goedel [5]).
Another comment. All planetary orbits are located in one and only one plane and every planet has a apsidal precession (perihel-rotation). Then the second focus of the planetary ellipses render circles around the sun. To ensure collision-free motion of the planets it is necessary that this circles are in the ellipses. Therefore the planetary ellipses are nearly circles.

7 Summary

To the cosmology. Suppose our solar system and the Milky Way (galaxy) are Schwarzschild-Spacetimes. This means inter alia, in each of these systems there is only one center of gravity, the sun in the center of the solar system and the black hole in the center the Milky Way, respectively. These centers of gravity are seen as ideal spherically symmetric and non-rotating. In a certain approximation, this are possible assumptions.
Then it follows, all the planets revolve in one plane around the sun and all suns circle in one plane around the black hole.
In reality, this is confirmed in a certain approximation. The planet planes of the solar system have only a low inclination to each other. The Milky Way in the night sky we perceive as a very flat disk-like formation. The objective appearance in nature confirm the results.

8 References

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